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VON NEUMANN-MORGENSTERN SOLUTIONS

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This paper gives a very short introduction to the von Neumann-Morgenstern theory of solutions, or stable sets, for multiperson cooperative games in characteristic function, or coalition, form. It includes the basic definitions which describes the essential notions for their n-person cooperative model, and it gives a very brief indication with references of some known results plus the general nature of this theory. This article was written to appear in the forthcoming book Handwörterbuch der mathematischen Wirtschaftswissenschaften published by Westdeutscher Verlag GmbH.

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PREFACE

This paper gives a very short introduction to the von Neumann-Morgenstern theory of solutions, or stable sets, for multiperson cooperative games in characteristic function, or coalition, form. It includes the basic definitions which describes the essential notions for their n-person cooperative model, and it gives a very brief indication with references of some known results plus the general nature of this theory. This article was written to appear in the forthcoming book Handwörterbuch der mathematischen Wirtschaftswissenschaften published by Westdeutscher Verlag GmbH.

VON NEUMANN-MORGENSTERN SOLUTIONS

In their famous book, Theory of Games and Economic Behavior, J. von Neumann and O. Morgenstern (1944) presented three fundamental models for analyzing games of strategy: the extensive (or tree) form, the normal (or strategic or "matrix") form, and the characteristic function (or coalition) form. This latter model, which comprises the greatest part of their text, is employed in the study of situations involving several participants acting in a cooperative mode. In such cooperative behavior the agents are allowed to get together in coalitions and to undertake joint action for the purpose of mutual gain.

The estimated worth of any potential coalition is a crucial aspect of such models, and this value is expressed as a numerical measure by what is called the characteristic function. The term von Neumann-Morgenstern solution refers to the final solution concept or end product for their coalition formulation of cooperative games. We will first give an intuitive verbal description of their model, and then a more precise technical definition.

There are four essential concepts or fundamental definitions which make up their basic cooperative model. First, a multiperson game is characterized by merely assigning a real number to each possible coalition of the n participants, where each number represents the value, wealth or power achievable by this coalition when its members cooperate. Next, one describes a set of n-dimensional payoff vectors called imputations. This presolution set represents all reasonable and realizable ways of distributing the available wealth among the n parties. Then, one introduces a preference relation between certain pairs of imputations. One imputation is said to dominate another, if there is some coalition in which each of its members prefers the former payoff to the latter one, and if this coalition as a whole is not obtaining more in the former imputation than it can effectively realize in the game through its own efforts. Finally, a von Neumann-Morgenstern solution is any subset of the imputation space which possesses a certain internal and external consistency; namely, no imputation in a solution dominates another one in this solution, and any imputation not in the solution is dominated by another one within the solution. So a solution when taken as a whole is preferred to precisely those imputations outside of this set. That is, a solution set is the complement of its "dominion." A more precise description and technical definition of this classical model follows.

An n-person game (in characteristic function form) consists of a pair (N,v) where $N = \{1,2,\dots,n\}$ is a set of n players labeled by $1,2,\dots,n$, and where v is a characteristic function which assigns a real number $v(S)$ to each nonempty subset (or coalition) S of N. One normally takes the integer $n \geq 3$, and assumes that v is superadditive, i.e., $v(S \cup T) \geq v(S) + v(T)$ for disjoint subsets S and T; but this latter assumption is not necessary for much of the theory. The set of imputations, denoted by A, consists of all vectors x such that $x_1 + x_2 + \dots + x_n = v(N)$ and $x_i \geq v(\{i\})$ for all i in N, where each x_i is a real number and represents a payoff to player i. These two relations are referred to as Pareto optimality and individual rationality, respectively. An imputation x dominates an imputation y whenever there is some coalition S such that $x_i > y_i$ for all i in S and $\sum_{i \in S} x_i \leq v(S)$. When this latter condition is satisfied one says that x is effective for S. A subset V of A is called a von Neumann-Morgenstern solution if it satisfies two conditions: no imputation x in V dominates any imputation in V, and every imputation y not in V is dominated by at least one imputation in V. These conditions are called internal stability and external stability, respectively, and now one frequently refers to such a solution as a stable set.

are also known, e.g., any solution contains the core and is in turn a subset of those imputations left undominated by the core.

1957, 1959, 1964). It turns out that many games have a great abundance of different solution sets. This lack of uniqueness, plus the great multiplicity of imputation in most individual solutions, leaves a great arbitrariness or ambiguity in determining any ultimate payoff vector for many games. Furthermore, some solutions are known to have a very complex or elaborate structure as illustrated by L. S. Shapley (1959, 1967), and it is unlikely that one will be able to give a plausible intuitive or behavioralistic interpretation to such irregular sets, or to design algorithmic methods for constructing them. It appears in general that solution theory is a deep and difficult mathematical subject and that many rich and fascinating structures are possible. A very readable introduction to solution theory as well as the most complete bibliography on the subject will appear in Chapter 6 of a forthcoming book by L. S. Shapley and M. Shubik; and a preliminary version by the authors (1973) has appeared in report form. A more technical survey of some important results on this subject was written by Lucas (1971).

Several generalizations, variations and extensions of the classical von Neumann-Morgenstern theory of solutions have also been investigated. These include in part: the games without side payments surveyed by R. J. Aumann (1967) and included in Shapley and Shubik (1973), the games in partition function form presented by R. M. Thrall and Lucas (1963), more dynamical approaches to solution concepts as discussed in the work of R. J. Weber (1974), the game models which make use of an infinite number of players, and models built upon other abstract mathematical structures. In addition, several different types of models and solution concepts along the lines of the classical formulation for games in coalition form have been introduced. The core, value theories, the bargaining sets and ψ-stability for cooperative games are discussed elsewhere in this volume. A few interrelations between the different theories are also known, e.g., any solution contains the core and is in turn a subset of those imputations left undominated by the core.

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